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Stringy sums and corrections to the quantum string Bethe ansatz

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Abstract: We analyze the effects of zeta-function regularization on the evaluation of quantum corrections to spinning strings. Previously, this method was applied in the $\mathfrak{sl}(2)$ subsector and yielded agreement to third order in perturbation theory with the quantum string Bethe ansatz. In this note we discuss related sums and compare zeta-function regularization against exact evaluation of the sums, thereby showing that the zeta-function regularized expression misses out perturbative as well as non-perturbative terms. In particular, this may imply corrections to the proposed quantum string Bethe equations. This also explains the previously observed discrepancy between the semi-classical string and the quantum string Bethe ansatz in the regime of large winding number.

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1. Introduction and summary

Explicit checks of the AdS/CFT correspondence beyond the supergravity approximation have been obstructed by the disjointness of the regimes in which gauge theory and string theory are understood in perturbation theory. Exact quantization of string theory on $AdS_5 \times S^5$ may help overcoming this problem and has therefore been the focus of much recent investigations. Key progress in this direction was triggered by the insight gained from studying the AdS/CFT correspondence in specific limits, as initiated by [1, 2], and in [3 – 9].

Further insight was obtained by identifying the integrable structures both in gauge and string theory. On the gauge theory side, this was deduced from the identification of the planar one-loop dilatation operator of $N = 4$ SYM with the hamiltonian of an integrable (super) spin chain [10, 11], solvable by means of a Bethe ansatz. The extension of the integrable structure to higher loops was subsequently shown in [12 – 14].[1] On the other hand, integrability of the string sigma model on $AdS_5 \times S^5$ [16] was observed in [17], and then utilised to test the AdS/CFT correspondence [18 – 20].[2] An important step linking the

[1] Altough integrability breaks down beyond the planar limit, some remnants of it persist and can be used to study decays of semi-classical strings [15].

[2] For reviews and further references see [21 – 25].

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two integrable structures on more general grounds was made in [26] by the construction of a set of Bethe equations for the classical string sigma-model.[3] These were then compared to the gauge theory Bethe equations in the thermodynamic limit, first for various subsectors and then the full $N = 4$ SYM and $AdS_5 \times S^5$ superstring [26, 32 − 36].

Inspired by the classical Bethe equations, a proposal was put forward for the description of quantum strings on $AdS_5 \times S^5$ [37 − 39]. It was conjectured that the string spectrum can be described by a new type of quantum string Bethe equations, which diagonalize some underlying string chain, and which are obtained by discretizing the classical string Bethe equations [26]. The conjectured quantum string Bethe equations were rigorously tested at infinite λ. However, they could potentially receive $1/\sqrt{\lambda}$ corrections [37].

To further test the proposal of [37 − 39], a detailed comparison between the one-loop worldsheet correction to the energy of a particular string configuration (which was computed semi-classically) to the finite size corrections following from the quantum string Bethe ansatz was recently performed [40]. The configuration studied was a circular string spinning in $AdS_3 \times S^1$ [41]. In this case the correction to the classical energy depends on two parameters $\mathcal{J}$ and k ($\mathcal{J}^2 = 1/\lambda' = J^2/\lambda$), where k is the string winding number and J is the spin in the S^1 direction. In [40] the comparison between semi-classical strings and Bethe ansatz was studied in the following two regimes: large $\mathcal{J}$ (and finite k) and large k (and finite $\mathcal{J}$).

In the first instance, due to the high complexity of the sums for the semi-classical string corrections, the analysis was performed by first expanding the summands in the parameter $1/\mathcal{J}$ (assuming that the summation index n is smaller than $\mathcal{J}$) and subsequent resummation. This procedure clearly breaks down for $n \geq \mathcal{J}$, and thus yields divergent expressions at each order in $1/\mathcal{J}^{2l}$. However upon zeta-function regularisation these agree with the Bethe ansatz in the first three orders in $1/\mathcal{J}^2$ [40]. This extended the leading order agreement previously found in [42, 43]. Other discussions of $1/J$ corrections have appeared in [18, 44 − 47].

In the second case of large winding number k, exact evaluation of the sum (which did not involve zeta-function regularization) resulted in a disagreement with the prediction of the string Bethe ansatz already at leading order in $1/k$ [40]. A similar mismatch was observed numerically.

As a possible explanation for the incompatibility of these results it was proposed that zeta-function regularization may not correctly sum the semi-classical string result [40]. A numerical analysis was performed to confirm this conjecture, but due to the insufficient numerical precision it was not possible to deduce a firm conclusion in its favour.

In this note we further examine this issue. We find strong evidence that zeta-function regularization does not give the correct answer for the sums in question. We first consider a simple toy example of a sum which has the same divergence problems when expanded in $1/\mathcal{J}$ as the sum in [40]. We then discuss the case of the folded string in the $\mathfrak{sl}(2)$ subsector and circular string in the $\mathfrak{su}(2)$ subsector [4, 5]. We evaluate the sums in question first

[3] See also [27 − 30] which identified the infinite tower of conserved charges on both sides. The classical string sigma-model reduces in the large spin limit to the effective action of the spin-chain, as was first observed in [31].

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by zeta-function regularization and then exactly, using various methods developed in [48, 49, 40]. These results confirm that zeta-function regularization does not reproduce the full sum. The explicit analysis (in the $\mathfrak{su}(2)$ subsector) shows that although the coefficients of $1/\mathcal{J}^{2n}$ in the expansion are correctly reproduced by the zeta-function regularisation, the coefficients of $1/\mathcal{J}^{2n+1}$ are not present, as well as the possibly non-vanishing non-perturbative contributions (i.e. of order $e^{-\mathcal{J}}$). Both types of terms do not follow from the quantum string Bethe equations, explaining thus the mismatch in the large k regime found in [40]. In particular the oscillatory behaviour observed in the large k limit in [40], is hidden in the exponential terms, which are entirely missed by zeta-function regularization.[4]

One important outcome of this analysis is that the terms in the string sums which are not captured by the quantum Bethe equations are non-analytic in the coupling, being proportional to $(\sqrt{\lambda'})^{2n+1}$ for integral n and $e^{-1/\sqrt{\lambda'}}$. It would be important to modify the S-matrix of [37 − 39] to incorporate these effects. Some of these issues are discussed in [50], where the terms with odd powers of $1/\mathcal{J}$ were also found in the $\mathfrak{su}(2)$ subsector and the relation to the Bethe ansätze in [37 − 39] was discussed.

The plan of this note is as follows. We first discuss two relatively simple sums (a toy model, as well as the folded string solution), which can be evaluated both exactly and by zeta-function regularization. In both cases zeta-function regularization fails to reproduce the exact sum. In section 4 we apply an approximation method, replacing the sum by an integral. Comparison with the exact expression for the sums, shows that the approximate evaluation correctly reproduces the terms missing in the zeta-function regularized result. We then apply this method to the $\mathfrak{su}(2)$ string and by comparing it with the zeta-function evaluated result, identify the missing terms.

2. Folded string solution

In this section we consider the one-loop energy shift for the folded rigid string, which rotates with a single spin S in AdS_3 and no spin in S^5. This correction was computed in [3], and is (in approximation) given by

$$\kappa \delta E_{\rm fold} = \sum_{n=1}^{\infty} \sqrt{n^2+4\kappa^2} + 2\sqrt{n^2+2\kappa^2} + 5n - 8\sqrt{n^2+\kappa^2}\,, \tag{2.1}$$

where $\kappa \sim \log \mathcal{S}$, $\mathcal{S} = S/\sqrt{\lambda}$. We wish to evalute this sum for large values of the parameter κ.[5] Recall, that the asymptotic value for the sum, obtained in [3] by replacing the sum with an integral is

$$\delta E^{FT}_{\rm fold} = -3\log 2\ \kappa + O(\kappa^0)\,. \tag{2.2}$$

In the following sections we shall evaluate the sum (2.1) first by naive zeta-function regularization and then by various exact evaluation methods. This will show that zeta-function fails to reproduce the correct sum.

[4] We are grateful to K. Zarembo for this remark.

[5] We thank A. Tseytlin for the suggestion to consider this sum.

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2.1 Zeta-function regularization

Let us first evaluate the sum along the lines of the zeta-function regularization applied in [40]. In order to do so, we pull the large-κ limit into the sum, i.e. expand each summand in $1/\kappa$ assuming that the summation index n is smaller than κ. This expansion is obviously incorrect when $n \sim \kappa$, which reflects itself in the divergence of the resulting sums at each order in $1/\kappa$ — despite the fact that the initial sum is convergent. We regularize these divergences using the zeta-function $\zeta(z)$ analytically continued to negative integers. This can in fact be done to all orders in $1/\kappa$ and results in

$$\begin{aligned}\delta E_{\text{fold}} &= \sum_n 2(\sqrt{2}-3) + \frac{1}{\kappa}\sum_n 5n + O\left(\frac{1}{\kappa^2}\right) \\ &= (3-\sqrt{2}) - \frac{5}{12}\frac{1}{\kappa} + O(e^{-\kappa}) .\end{aligned} \tag{2.3}$$

Here we used that $\zeta(-1) = -B_2/2 = -1/12$ and each higher term is a sum over n^{2l}, and thus vanishes in the zeta-function prescription. This clearly contradicts the asymptotics in (2.2) by missing out the crucial linear term in κ. The result (2.2) was obtained by an approximative method, so it would be desirable to have independent checks of the sum to confirm the failure of zeta-function regularization. We shall subsequently present three methods which will be in agreement with (2.2), as well as produce subleading terms obtained in (2.3) (up to exponentially small corrections).

2.2 Asymptotic evaluation

A method to asymptotically evaluate sums of the type (2.1) was obtained in [48, appendix B] in the context of plane-wave string field theory. The main idea is to represent the square root terms using the integral representation of the Gamma-function

$$\frac{1}{x^z} = \frac{1}{\Gamma(z)}\int_0^\infty dt t^{z-1} e^{-xt} , \tag{2.4}$$

which is valid for $x, z > 0$. For this to be applicable, we first act with $\frac{\partial}{\partial\kappa}\left(\frac{1}{\kappa}\frac{\partial}{\partial\kappa}\right)$ on the sum (2.1), which reduces to the expression

$$R = -8\kappa \sum_{n=1}^{\infty}\left(\frac{2}{(n^2+4\kappa^2)^{3/2}} + \frac{1}{(n^2+2\kappa^2)^{3/2}} - \frac{1}{(n^2+\kappa^2)^{3/2}}\right) . \tag{2.5}$$

Each partial sum is now absolutely convergent and can be asymptotically evaluated separately using (2.4). The relevant asymptotics derived in [48][6] are

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(\mathcal{J}^2+n^2)^{3/2}} &= \frac{2}{\sqrt{\pi}\mathcal{J}^3}\int_0^\infty ds s^{1/2} e^{-s}\left(\theta\left(\frac{s}{(\pi\mathcal{J}^2)}\right) - 1\right) \\ &= \frac{1}{\mathcal{J}^2} - \frac{1}{2\mathcal{J}^3} + O\left(e^{-\mathcal{J}}\right) .\end{aligned} \tag{2.6}$$

[6]Similar sums are discussed in [51–53].

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Here $\theta(t)=\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$ and we modular transformed and used the asymptotics $\theta(t)\to 1$ as $t\to\infty$. Applied to the present case we obtain

$$R=\frac{1}{\kappa^2}(3-\sqrt{2})+O(e^{-\kappa})\,, \tag{2.7}$$

which after repeated integration results in

$$\delta E_{\text{fold}}=(3-\sqrt{2})+\frac{c_1\kappa}{2}+\frac{c_0}{\kappa}+O(e^{-\kappa})\,, \tag{2.8}$$

where c_i are integration constants, which need to be determined in some other way. In particular, this is in accord with [3], as there are choices for c_i, for which the sums can be made to agree. The integration constants can be derived in the way done in [49], but we shall present two alternative methods to compute the sum exactly.

2.3 Bessel function evaluation

The energy shift can be likewise evaluated using the following integral representation obtained in [40, eq. (2.7) and (2.10)]. Recall that

$$\sum_{n=1}^{\infty}\left(\sqrt{(n+\gamma)^2+\alpha^2}+\sqrt{(n-\gamma)^2+\alpha^2}-2n-\frac{\alpha^2}{n}\right)=\gamma^2-\sqrt{\gamma^2+\alpha^2}+F(\{\gamma\},\alpha)\,, \tag{2.9}$$

where we defined the function

$$F(\beta,\alpha)\equiv\sqrt{\alpha^2+\beta^2}-\beta^2+\alpha^2\int_0^{\infty}\frac{d\xi}{e^{\xi}-1}\left(\frac{2J_1(\alpha\xi)}{\alpha\xi}\cosh\beta\xi-1\right). \tag{2.10}$$

For large α the asymptotic behaviour of this function is

$$F(\beta,\alpha)=-\alpha^2\ln\left(\frac{e^{C-1/2}}{2}\alpha\right)+\frac{1}{6}+O\left(e^{-\alpha}\right), \tag{2.11}$$

where $C=0.5772\ldots$ is the Euler constant. Applying this to (2.1) results in

$$\delta E_{\text{fold}}=-3\ln 2\,\kappa+3-\sqrt{2}-\frac{5}{12\kappa}+O(e^{-\kappa})\,, \tag{2.12}$$

in agreement with [3] and implying that the integration constants in (2.8) are $c_0=-5/12$ and $c_1=-6\log(2)$. Note that this also calculates all subleading terms up to exponential (powerlike in $1/\mathcal{S}$ as $\kappa\sim\log\mathcal{S}$) corrections.

2.4 Generalized zeta-function evaluation

The result obtained with Bessel functions in the last subsection can be confirmed by the following analytic continuation argument. Consider a generalization of the Riemann zeta-function

$$\zeta(s,\kappa)=\sum_{n=1}^{\infty}\frac{1}{(n^2+\kappa^2)^s}\,. \tag{2.13}$$

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This is to begin with not well-defined for the choice $s = -1/2$ that we are interested in, but the generalized zeta function can be analytically continued to this value. Again, representing the summand using the Gamma function integral representation as (2.6) derived in appendix B of [48], it follows that the large κ asymptotics of this expression is

$$\zeta(s,\kappa) = -\frac{1}{2}\kappa^{2s} + \frac{1}{2\kappa^{2s-1}} \frac{\Gamma(1/2)\Gamma(s-1/2)}{\Gamma(s)} + O(e^{-\kappa}) \,. \tag{2.14}$$

Note now, that this would have been obtained likewise by approximating the sum by an integral, namely setting $u = n/\kappa$ in the large κ limit

$$\zeta(s,\kappa) \sim \frac{1}{\kappa^{2s-1}} \int_0^\infty du \frac{1}{(1+u^2)^s} = \frac{1}{2\kappa^{2s-1}} \frac{\Gamma(1/2)\Gamma(s-1/2)}{\Gamma(s)} \,. \tag{2.15}$$

Applying this to $\delta E_{\rm fold}$ for $s = -1/2 + \alpha$ for $\alpha \to 0$ and that the Riemann zeta-function analytically continued gives $\zeta(-1) = -1/12$, we arrive at

$$\delta E_{\rm fold} = -3 \log 2 \;\kappa + (3 - \sqrt{2}) - \frac{5}{12\kappa} + O(e^{-\kappa}) \,, \tag{2.16}$$

in agreement with the above Bessel function evaluation and [3].

This method is quite general and also explains why zeta-function regularization does not always work. Namely, zeta-function regularization drops the term that comes from the Gamma-functions in (2.14).

2.5 Exponential corrections

So far we have refrained from working out explictly the exponential corrections at $O(e^{-\kappa})$. These may however turn out to be crucial for comparison to the quantum string Bethe ansatz. We shall now prove that in the simpler case of the folded string these terms are indeed non-vanishing and find explicit formulas for these terms. As the starting point, consider the asymptotic evaluation method presented earlier. Recall that

$$\kappa \sum_{n=1}^\infty \frac{1}{(n^2+\kappa^2 a^2)^{3/2}} = -\frac{1}{2a^3\kappa^2} + \frac{1}{a^2\kappa} + \frac{2}{a^2\kappa} \int_0^\infty dt e^{-t} \sum_{n=1}^\infty \left(e^{-\pi^2 n^2 \kappa^2 a^2/t} \right) . \tag{2.17}$$

The last term is the exponential correction term and can be further evaluated

$$\begin{aligned} R_{\rm exp} &= \frac{2}{a^2\kappa} \sum_{n=1}^\infty \int_0^\infty dt e^{-t-\pi^2 n^2 a^2 \kappa^2/t} \\ &= \frac{4}{a^2} \sum_{n=1}^\infty 2\pi a n K_1(2\pi n a \kappa) \,. \end{aligned} \tag{2.18}$$

Note that $\partial_\kappa K_0(2\pi n a \kappa) = -2\pi n a K_1(2\pi n a \kappa)$. So, already integrating up once with respect to κ yields

$$\int d\kappa R_{\rm exp} = -\frac{4}{a^2} \sum_{n=1}^\infty K_0(2\pi n a \kappa) \,. \tag{2.19}$$

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Then apply the integral represetation (see also [49, appendix D])

$$K_0(z\kappa) = \int_0^\infty dt \frac{e^{-z\sqrt{t^2+\mu^2}}}{\sqrt{t^2+\mu^2}}\,, \tag{2.20}$$

and perform the sum, which yields

$$\begin{aligned}\kappa \int d\kappa R_{\rm exp} &= -\frac{4}{a^2}\kappa \int_0^\infty dt \frac{1}{\sqrt{t^2+\kappa^2}} \frac{1}{e^{2\pi a\sqrt{t^2+\kappa^2}}-1} \\ &= -\frac{2}{a^2}\kappa \int_1^\infty dr \frac{\coth(a\kappa\pi r)-1}{\sqrt{s^2-1}}\,. \end{aligned} \tag{2.21}$$

Integrating repeatedly with respect to κ, we arrive at

$$\int d\kappa\kappa \int d\kappa R_{\rm exp} = -\frac{2}{a^2}\int_1^\infty dr \left[\frac{\kappa\log\left(1-e^{-2\pi a\kappa r}\right)}{ar\sqrt{r^2-1}} + \frac{(r^3+r^2-1){\rm Li}_2\left(e^{-2\pi a\kappa r}\right)}{2a^2\pi^2 r^2\sqrt{r^2-1}}\right]. \tag{2.22}$$

This is a closed formula for the exponential correction term we were looking for. Adding up the contributions with the various choices for a of each summand in (2.1) produces the complete correction term for the folded string.

If one is interested in obtaining the first correction term in $e^{-\kappa}O(\kappa^0)$ explicitly, one can proceed as follows. Note that $\int d\kappa\kappa K_0(b\kappa) = -\kappa K_1(b\kappa)/b$. So we obtain

$$\int d\kappa\kappa \int d\kappa R_{\rm exp} = \frac{2\kappa}{\pi a^3}\sum_{n=1}^\infty \frac{K_1(2\pi n a\kappa)}{n}\,. \tag{2.23}$$

With the asymptotics $K_1(z) = \sqrt{\pi/2z}e^{-z}(1+O(1/z))$ we obtain that the first exponential correction terms are

$$\int d\kappa\kappa \int d\kappa R_{\rm exp} = \frac{\kappa}{\pi a^3}\sum_{n=1}^\infty e^{-2\pi n a\kappa}\frac{1}{n}\sqrt{\frac{1}{na\kappa}}\left[1+O\left(\frac{1}{\kappa}\right)\right]. \tag{2.24}$$

Adding together the terms with the correct prefactors and choices for a gives the correction to (2.1).

In summary we have shown in this section that the exponential corrections do not vanish for the folded string case. It would of course be interesting to see, whether they contribute in more complicated sums than (2.1), such as the one-loop energy shift for the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ subsectors.

3. Toy model

As a second test case consider the situation of two bosonic and two fermionic frequencies with the energy shift given by

$$\delta E_{\rm toy} = \sum_{n=1}^\infty \sqrt{1+\frac{(n+\gamma)^2}{\mathcal{J}^2}} + \sqrt{1+\frac{(n-\gamma)^2}{\mathcal{J}^2}} - 2\sqrt{1+\frac{n^2}{\mathcal{J}^2}}\,, \tag{3.1}$$

where γ is a constant independent of $\mathcal{J}$ and the sum is convergent in the same sense as for the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ spinning strings. Again we compare zeta function regularization with the exact evaluation of the sum in the large $\mathcal{J}$ limit and find disagreement.

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3.1 Zeta-function regularization

For the naive perturbative evaluation of (3.1), pull the large $\mathcal{J}$ limit through the sum. As each term in the $1/\mathcal{J}$ expansion is of order n^0 or higher, using zeta-function regularization the sum evaluates to

$$\delta E^{\zeta}_{\text{toy}} = -\frac{1}{2}\left(-2 + 2\sqrt{1+\frac{\gamma^2}{\mathcal{J}^2}}\right). \tag{3.2}$$

Expanding this in $1/\mathcal{J}$ yields the energy shift at arbitrary loop orders as obtained from this prescription.

3.2 Asymptotic evaluation of sums

Alternatively, in this simple case, one can evaluate the sum exactly (up to terms $e^{-\mathcal{J}}$) using the method in [48]. Consider the sum

$$\delta E_{\text{toy}}\mathcal{J} = S = \sum_{n=1}^{\infty} \sqrt{(n+\gamma)^2+\mathcal{J}^2} + \sqrt{(n-\gamma)^2+\mathcal{J}^2} - 2\sqrt{n^2+\mathcal{J}^2}\,. \tag{3.3}$$

Then following the strategy in [48], act with $\frac{\partial}{\partial\mathcal{J}}\left(\frac{1}{\mathcal{J}}\frac{\partial}{\partial\mathcal{J}}\right)$ to obtain

$$R = -\mathcal{J}\sum_{n=1}^{\infty}\frac{1}{((n+\gamma)^2+\mathcal{J}^2)^{3/2}} + \frac{1}{((n-\gamma)^2+\mathcal{J}^2)^{3/2}} - 2\frac{1}{(\mathcal{J}^2+n^2)^{3/2}}\,. \tag{3.4}$$

Now each part of the sum is absolutely convergent by itself and can be evaluated and later on integrated up to give the result for the complete sum. The last summand is easiest and is evaluated the same way as in [48, appendix B], i.e. (2.6). The remaining two terms are computed likewise. First recall the definition of the generalized theta-functions

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(t) = \sum_{n=-\infty}^{\infty} e^{\pi t(n+a)^2+2\pi nbi}\,, \tag{3.5}$$

which satisfies the modular transformation law, shown by Poisson resummation,

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(t) = \frac{1}{\sqrt{-t}}\theta\begin{bmatrix} b \\ -a \end{bmatrix}(1/t)\,. \tag{3.6}$$

So in particular we can write

$$\theta\begin{bmatrix} \gamma \\ 0 \end{bmatrix}(-t/\pi) = e^{-\gamma^2 t} + \sum_{n=1}^{\infty}\left(e^{-(n+\gamma)^2 t} + e^{-(n-\gamma)^2 t}\right)\,. \tag{3.7}$$

This allows the evaluation of the remaining two terms in the sum, again asymptotically for

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large $\mathcal{J}$

$$
\begin{aligned}
\sum_{n=1}^{\infty} &\frac{1}{((n+\gamma)^2+\mathcal{J}^2)^s}+\frac{1}{((n-\gamma)^2+\mathcal{J}^2)^s}= \\
&=\frac{1}{\Gamma(s)}\int_0^\infty dr r^{s-1}e^{-\mathcal{J}^2 r}\sum_{n=1}^{\infty}\left(e^{-(n+\gamma)^2 r}+e^{-(n-\gamma)^2 r}\right) \\
&=\frac{1}{\Gamma(s)\mathcal{J}^{2s}}\int_0^\infty dt t^{s-1}e^{-t}\left(\theta\begin{bmatrix}\gamma\\0\end{bmatrix}(-t/(\pi\mathcal{J}^2))-e^{-\gamma^2 t/\mathcal{J}^2}\right) \\
&=-\frac{1}{(\mathcal{J}^2+\gamma^2)^s}+\frac{\sqrt{\pi}}{\Gamma(s)\mathcal{J}^{2s-1}}\int_0^\infty dt t^{s-3/2}e^{-t}\theta\begin{bmatrix}0\\-\gamma\end{bmatrix}(-\pi\mathcal{J}^2/t) \qquad (3.8)\\
&=-\frac{1}{(\mathcal{J}^2+\gamma^2)^s}+\frac{\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s)\mathcal{J}^{2s-1}}+\frac{\sqrt{\pi}}{\mathcal{J}^{2s-1}}\int_0^\infty dt t^{s-3/2}e^{-t}\left(\theta\begin{bmatrix}0\\-\gamma\end{bmatrix}(-\pi\mathcal{J}^2/t)-1\right).
\end{aligned}
$$

For $s=3/2$ the last term is of order $e^{-\mathcal{J}}$, which can be seen by changing to $u=\mathcal{J}^N t$. So in summary we obtain

$$
\sum_{n=1}^{\infty}\frac{1}{((n+\gamma)^2+\mathcal{J}^2)^{3/2}}+\frac{1}{((n-\gamma)^2+\mathcal{J}^2)^{3/2}}=\frac{2}{\mathcal{J}^2}-\frac{1}{(\mathcal{J}^2+\gamma^2)^{3/2}}+O\left(e^{-\mathcal{J}}\right). \qquad (3.9)
$$

Thus we obtain that

$$
R=-\mathcal{J}\left(\frac{1}{\mathcal{J}^3}-\frac{1}{(\mathcal{J}^2+\gamma^2)^{3/2}}\right). \qquad (3.10)
$$

Integrating up, we obtain

$$
\delta E_{\text{toy}}=\frac{1}{\mathcal{J}}\left(\mathcal{J}-\sqrt{\gamma^2+\mathcal{J}^2}\right)+c_0\mathcal{J}+\frac{c_1}{\mathcal{J}}+O(e^{-\mathcal{J}}), \qquad (3.11)
$$

which for vanishing integration constants agrees up to terms $O(e^{-\mathcal{J}})$ with the perturbative zeta-function regularized expression δE^{ζ}.

In order to determine the integration constants, differentiate with respect to γ and then evaluate the large $\mathcal{J}$ in analogy to [49]. However, we shall determine these using the Bessel and generalized zeta-function methods introduced earlier.

3.3 Bessel function evaluation

Consider now the evaluation using Bessel functions. First split the sum into two partial sums which both converge absolutely

$$
\begin{aligned}
\delta E_{\text{toy}}\mathcal{J} &= S = S_1+S_2 \\
S_1 &= \sum_{n=1}^{\infty}\sqrt{(n+\gamma)^2+\mathcal{J}^2}+\sqrt{(n-\gamma)^2+\mathcal{J}^2}-2n-\frac{\mathcal{J}^2}{n} \\
S_2 &= -2\sqrt{n^2+\mathcal{J}^2}+2n+\frac{\mathcal{J}^2}{n}. \qquad (3.12)
\end{aligned}
$$

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The representation (2.10) implies

$$\begin{aligned} S_1 &= \gamma^2 - \sqrt{\gamma^2 + \mathcal{J}^2} + F(\{\gamma\}, \mathcal{J}) \\ S_2 &= \mathcal{J} - F(0, \mathcal{J})\,. \end{aligned} \tag{3.13}$$

The large $\mathcal{J}$ asymptotics follow from (2.11), so that

$$\begin{aligned} S_1 &= \gamma^2 - \sqrt{\gamma^2 + \mathcal{J}^2} - \mathcal{J}^2 \ln \mathcal{J} - \mathcal{J}^2 \ln\left(\frac{e^{C-\frac{1}{2}}}{2}\right) + \frac{1}{6} + O\left(e^{-\mathcal{J}}\right) \\ S_2 &= \mathcal{J} + \mathcal{J}^2 \ln \mathcal{J} + \mathcal{J}^2 \ln\left(\frac{e^{C-\frac{1}{2}}}{2}\right) - \frac{1}{6} + O\left(e^{-\mathcal{J}}\right)\,, \end{aligned} \tag{3.14}$$

and thus the asymptotic expansion for the energy is up to exponentially small corrections

$$\delta E_{\mathrm{toy}} = \frac{1}{\mathcal{J}}\left(\gamma^2 + \mathcal{J} - \sqrt{\gamma^2 + \mathcal{J}^2}\right) + O\left(e^{-\mathcal{J}}\right). \tag{3.15}$$

This is in agreement with the asymptotic evaluation and determines the integration constants as $c_0 = 0$ and $c_1 = \gamma^2$.

3.4 Generalized zeta-function evaluation

To confirm the result from the last section, we apply analytic continuation to the following generalized zeta-function

$$\zeta(s, \gamma, \mathcal{J}) = \sum_{n=1}^{\infty} \frac{1}{((n+\gamma)^2 + \mathcal{J}^2)^s}\,. \tag{3.16}$$

Then by analytic continuation to $s = -1/2$ we can compute the sums in δE. The asymptotics for large values of $\mathcal{J}$ follow using (3.8) in the last section using generalized theta functions and setting $s = -1/2$

$$\zeta(s, \gamma, \mathcal{J}) + \zeta(s, -\gamma, \mathcal{J}) = -\frac{1}{(\gamma^2 + \mathcal{J}^2)^s} + \frac{\sqrt{\pi}\Gamma(s - 1/2)}{\Gamma(s)\mathcal{J}^{2s-1}} + \gamma^2 + O(e^{-\mathcal{J}})\,. \tag{3.17}$$

The last term in (3.8) for $s = -1/2$ is not exponentially suppressed and is extracted by performing the integral yielding $\sum a_n/(n\mathcal{J})^2 K_1(\pi\mathcal{J}n)$, which has the given asymptotics. Up to exponential corrections we obtain that the sum has large $\mathcal{J}$ behaviour given by

$$\begin{aligned} S &= \lim_{\alpha\to 0}\left\{\zeta\left(-\frac{1}{2}+\alpha, \gamma, \mathcal{J}\right) + \zeta\left(-\frac{1}{2}+\alpha, -\gamma, \mathcal{J}\right) - 2\zeta\left(-\frac{1}{2}+\alpha, 0, \mathcal{J}\right)\right\} \\ &= \lim_{\alpha\to 0}\left\{-(\gamma^2+\mathcal{J}^2)^{1/2-\alpha} + \frac{\sqrt{\pi}\Gamma(-1+\alpha)}{\Gamma(-1/2)}\mathcal{J}^2 + \gamma^2 - 2\left(-\frac{1}{2\mathcal{J}} + \frac{\mathcal{J}^2}{2}\frac{\Gamma(1/2)\Gamma(-1+\alpha)}{\Gamma(-1/2+\alpha)}\right)\right\} \\ &= \gamma^2 + \mathcal{J} - \sqrt{\gamma^2 + \mathcal{J}^2}\,. \end{aligned} \tag{3.18}$$

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This is again in agreement with the two independent methods of evaluation presented earlier and confirms the incompleteness of the evaluation by means of zeta-function regularization.

4. Zeta-function regularization versus exact summation

In the previous sections we have performed exact, analytic evaluations of the sums (2.1) and (3.1) using several methods. These were compared to the zeta function regularized expressions (2.3) and (3.2) and were found to disagree with them. We would now like to determine the origin of this disagreement.[7] The nature of this section is more experimental and it would be important to understand this in full generality, e.g. in relation with the observation in (2.15). In particular, it should be possible to extend this to the case of the $\mathfrak{sl}(2)$ subsector.

To proceed, we split the infinite sum into a finite sum, where zeta-function regularization applies, and another part, which will be approximated by simply replacing the sum by an integral. The correction terms that are computed by the Euler-Maclaurin summation formula will be discussed below. More precisely

$$\begin{aligned} S(\eta) &= \sum_{n=1}^{K} f(n,\eta) + \sum_{n=K}^{\infty} f(n,\eta) \\ &= S^{\mathrm{I}}(K,\eta) + S^{\mathrm{II}}(K,\eta)\,, \qquad K \gg 1\,, \end{aligned} \tag{4.1}$$

where we have denoted the large parameters κ and $\mathcal{J}$ in (2.1) and (3.1) by η. Since $K \gg 1$ the second sum $S^{\mathrm{II}}(\eta)$ can be replaced with an integral, which will be denoted by $\tilde{S}^{\mathrm{II}}$. Further let us assume that

$$1 \ll K \ll \eta\,. \tag{4.2}$$

Then the second sum (i.e. integral) $\tilde{S}^{\mathrm{II}}(\eta)$ can be expanded in $1/\eta$.

On the other hand, for the zeta-function regularization used in [40] one first expands $f(n,\eta)$ in $1/\eta$ and then resums the expanded series. It is clear that this expansion fails, when $n \sim \eta$, inducing spurious divergences. These were cured by introducing the zeta-function regularization, which effectively means that one multiplies all terms in the sum with a factor $e^{-\alpha n}$. Since $n \leq K$ in the first sum, the expansion in $1/\eta$ is correct one, and zeta function regularization does not affect this part of the result.

We thus focus only on the second sum. To compare the zeta-function regularized results with the integrated sum $\tilde{S}^{\mathrm{II}}$, we first need to determine the value of the zeta function that is cut-off K dependent, and approximated by the integral as when evaluating the sum S^{II}. For this, we use simply the replacement of the sum by an integral, as in [3]. More precisely, we use the right colum of the following equation as the values of the zeta-function (taking

[7] Some of the ideas in this section arose in discussions with A. Tseytlin. Similar observations have recently appeared in [50].

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$\alpha < 1/K$)

$$\begin{aligned}
\sum_{n=K}^{\infty} e^{-\alpha n} &= \frac{1}{\alpha} + \left(\frac{1}{2} - K\right) + O(\alpha) &\rightarrow \quad \int_K^{\infty} dn\, e^{-\alpha n} &= \frac{1}{\alpha} - K + O(\alpha) \\
\sum_{n=K}^{\infty} e^{-\alpha n} n &= \frac{1}{\alpha^2} + \left(-\frac{1}{12} + \frac{K}{2} - \frac{K^2}{2}\right) + O(\alpha) &\rightarrow \quad \int_K^{\infty} dn\, e^{-\alpha n} n &= \frac{1}{\alpha^2} - \frac{K^2}{2} + O(\alpha) \\
\sum_{n=K}^{\infty} e^{-\alpha n} n^2 &= \frac{2}{\alpha^3} - \frac{1}{6}\left(K - 3K^2 + 2K^3\right) + O(\alpha) &\rightarrow \quad \int_K^{\infty} dn\, e^{-\alpha n} n^2 &= \frac{2}{\alpha^3} - \frac{K^3}{3} + O(\alpha)\,.
\end{aligned} \tag{4.3}$$

Comparing to the standard Euler-Maclaurin summation formula yields that all extra tail and boundary terms contribute subleading in K and can be neglected. However, we will see that this heuristic method reproduces precisely the missing terms in the zeta-function regularization. We shall now compare the standard zeta-function regularized result with the integral version zeta-function regularized expression using this prescription. Let us first apply both methods to compute the sum $S^{\rm II}$ for the folded string (2.1) and the toy model (3.1).

4.1 Folded string and toy model

Approximating the sums (3.1), (2.1) with an integral, and subsequently expanding in $1/\eta$, we obtain, respectively

$$\begin{aligned}
S^{\rm II}_{\rm toy}(K, \mathcal{J}) &= \gamma^2 - K\gamma^2 \frac{1}{\mathcal{J}} + \left(\frac{1}{2}K^3\gamma^2 + \frac{1}{4}K\gamma^4\right)\frac{1}{\mathcal{J}^3} + O\left(\frac{K}{\mathcal{J}^5}\right) \\
S^{\rm II}_{\rm fold}(K, \kappa) &= -3\log 2\,\kappa^2 - 2(\sqrt{2}-3)K\kappa - \frac{5}{2}K^2 - \frac{1}{6}\left(\sqrt{2} - \frac{15}{2}\right)\frac{K^3}{\kappa} + O\left(\frac{K^5}{\kappa^3}\right)
\end{aligned} \tag{4.4}$$

On the other hand, expanding the summands $f(n,\eta)$ as done for the zeta-function regularization leads to

$$\begin{aligned}
f_{\rm toy}(n, \mathcal{J}) &= \gamma^2 \frac{1}{\mathcal{J}} + \left(-\frac{3}{2}n^2\gamma^2 - \frac{\gamma^4}{4}\right)\frac{1}{\mathcal{J}^3} + O\left(\frac{1}{\mathcal{J}^4}\right) \\
f_{\rm fold}(n, \kappa) &= 2(\sqrt{2}-3)\kappa + 5n + \left(\frac{\sqrt{2}}{2} - \frac{15}{4}\right) n^2 \frac{1}{\kappa} + O\left(\frac{1}{\kappa^2}\right).
\end{aligned} \tag{4.5}$$

Comparing the expansions (4.4) with (4.5), we note the absence of the leading, $1/\mathcal{J}^0$ and κ^2 terms in the expansion of the summands. Summing up the expanded terms (4.5) from (K, ∞) and using the zeta function results (4.3) we obtain the same results as in (4.4) except for the $1/\mathcal{J}^0$ and κ^2 terms, which were absent from the beginning in the expansion. These terms, being cut-off K independent parts of the sums, can be obtaind by setting $K = 0$ in the integral. So the difference between the two results is given by

$$\Delta(\eta) = \int_0^{\infty} f(n,\eta)\, dn\,. \tag{4.6}$$

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4.2 The circular string in the $\mathfrak{su}(2)$ subsector

In this section we will consider the evaluation of the 1-loop energy energy shift corresponding to the circular string which rotates in an S^3 inside the S^5 with two equal spins $J_1 = J_2 = J/2$. The energy shift takes the following form [5, 9, 42]

$$\delta E = \delta E^{(0)} + \sum_{n=1}^{\infty} \delta E^{(n)} \,, \tag{4.7}$$

where

$$\begin{aligned} \delta E^{(0)} &= 2 + \sqrt{1 - \frac{2k^2}{\mathcal{J}^2 + k^2}} - 3\sqrt{1 - \frac{k^2}{\mathcal{J}^2 + k^2}} \\ \delta E^{(n)} &= 2\sqrt{1 + \frac{(n + \sqrt{n^2 - 4k^2})^2}{4(\mathcal{J}^2 + k^2)}} + 2\sqrt{1 + \frac{n^2 - 2k^2}{\mathcal{J}^2 + k^2}} + 4\sqrt{1 + \frac{n^2}{\mathcal{J}^2 + k^2}} - \\ &\quad - 8\sqrt{1 + \frac{n^2 - k^2}{\mathcal{J}^2 + k^2}} \,. \end{aligned} \tag{4.8}$$

The zeta-function regularized version of the sum is derived to all orders in $1/\mathcal{J}$ in the appendix. It is hard to exactly repeat the procedure from the previous section for the sum (4.8) due to the complexity of the integral $\tilde{S}^{\mathrm{II}}$. So let us instead first expand the sum (4.8) in the small parameter k and then repeat the computation from the previous section order by order in k. Note also, that although the winding number k is in principle integer valued, in the regime which we are interested, namely $\mathcal{J} \gg 1$, $n > K \gg 1$, the expansion in small k is justified.

The expansion of the summand (4.8) is

$$\begin{aligned} \delta E^{(n)} &= -\frac{(\mathcal{J}^2 + 2n^2)}{\mathcal{J} n^2 (\mathcal{J}^2 + n^2)^{3/2}} k^4 + \frac{-2\mathcal{J}^4 - 2\mathcal{J}^2 n^2 + n^4}{\mathcal{J}^3 n^4 (n^2 + \mathcal{J}^2)^{3/2}} k^6 + O(k^8) \\ &\equiv \delta E_1^{(n)} k^4 + \delta E_2^{(n)} k^6 + O(k^8) \,. \end{aligned} \tag{4.9}$$

We can now repeat the procedure from the previous section for the sums δE_1 and δE_2.

Expansion of the first integral yields

$$\int_K^{\infty} dn\, \delta E_1^{(n)} = -\frac{1}{\mathcal{J} K \sqrt{\mathcal{J}^2 + K^2}} = -\frac{1}{K}\frac{1}{\mathcal{J}^2} + \frac{1}{2} K \frac{1}{\mathcal{J}^4} - \frac{3}{8} K^3 \frac{1}{\mathcal{J}^6} + O\left(\frac{1}{\mathcal{J}^8}\right). \tag{4.10}$$

The integrated function thus admits an integer power expansion in $\frac{1}{\mathcal{J}^{2n}}$, and thus is analytic in λ'. On the other hand, the naive expansion (i.e. the expansion where we assume that $n < \mathcal{J}$) of the integrand $\delta E_1^{(n)}$ gives

$$\delta E_1^{(n)} = -\frac{1}{n^2}\frac{1}{\mathcal{J}^2} - \frac{1}{2}\frac{1}{\mathcal{J}^4} + \frac{9}{8} n^2 \frac{1}{\mathcal{J}^6} + O\left(\frac{1}{\mathcal{J}^8}\right). \tag{4.11}$$

As expected, these terms yield divergent sums starting from $1/\mathcal{J}^4$, however they appear with powers $1/\mathcal{J}^{2k}$, i.e. the same powers of the expansion in (4.10). Integrating the expression (4.11) and using the integral version of the zeta-function prescription (4.3), we reproduce all terms in (4.10).

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The evaluation of the second order term $\delta E_2^{(n)}$ is different

$$\int_K^\infty dn\,\delta E_2^{(n)} = \frac{-2\mathcal{J}^4 + 2\mathcal{J}^2K^2 + K^3(K - \sqrt{\mathcal{J}^2+K^2})}{3\mathcal{J}^5K^3\sqrt{\mathcal{J}^2+K^2}}$$
$$= -\frac{2}{3}\frac{1}{K^3}\frac{1}{\mathcal{J}^2} + \frac{1}{K}\frac{1}{\mathcal{J}^4} - \frac{1}{3}\frac{1}{\mathcal{J}^5} - \frac{1}{4}K\frac{1}{\mathcal{J}^6} + O\left(\frac{1}{\mathcal{J}^9}\right). \tag{4.12}$$

The main difference to the former case is, the presence of the term $1/\mathcal{J}^5$, which is non-analytic in λ' and which appears as the cuf-off K independent part of the integral. On the other hand, the naive expansion of $\delta E_2^{(n)}$ yields

$$\delta E_2^{(n)} = -\frac{1}{n^4}\frac{1}{\mathcal{J}^2} + \frac{1}{n^2}\frac{1}{\mathcal{J}^4} + \frac{1}{4}\frac{1}{\mathcal{J}^6} - \frac{7}{8}n^2\frac{1}{\mathcal{J}^8} + O\left(\frac{1}{\mathcal{J}^{10}}\right), \tag{4.13}$$

where all terms are analytic in λ'. Since the zeta-function prescription does not change the order in $1/\mathcal{J}$ in the expansion, it is thus clear that the terms at order $1/\mathcal{J}^5$ in (4.12) can never be reproduced by the zeta-function regularization of the expression (4.13). The regular terms in (4.12) are on the other hand easily reproduced using the cut-off zeta-function regularization (4.3). Similar analysis for the order k^6 and higher, yields the discrepancy between the zeta-function regularization and the exact string result at the orders $1/\mathcal{J}^{2k+1}$.

A more detailed analysis of the correction terms in the Euler-Maclaurin summation formula for the sums appearing in (4.9), shows that only the coefficients of $1/\mathcal{J}^{2k}$ are corrected, and also that all these corrections are supressed with inverse powers of the cutoff. Thus, the approximate integral evaluation of the coefficients of $1/\mathcal{J}^{2k+1}$ gives the exact result.[8] It should be possible to resum the effect of these terms that are missed by zeta-function regularization.

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A. Zeta-function regularization for $\mathfrak{su}(2)$

In this appendix we derive the all orders result that follows from zeta-function regularization in the $\mathfrak{su}(2)$ subsector, where the energy shift is (4.8). Up to two-loops the energy shift has appeared recently in [47]. Evaluating the sum perturbatively in $1/\mathcal{J}^2$, i.e., $\delta E = \sum_{i=0}^\infty \delta E_i/\mathcal{J}^{2i}$, the energy shifts at the first three loop orders are as follows.

- **1-loop:**

$$\delta E_1 = \frac{k^2}{2} + \frac{1}{2}\sum_n \left(2k^2 - n^2 + n\sqrt{n^2-4k^2}\right). \tag{A.1}$$

[8] Recall that the sum S^I in (4.1) only contributes to the even powers of $\mathcal{J}$.

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• **2-loop:**

$$\delta E_2 = -\frac{5k^4}{8} + \frac{1}{8}\sum_n \left(-10k^4 + n^4 - (n^2+2k^2)n\sqrt{n^2-4k^2}\right). \qquad \text{(A.2)}$$

At large n the sum has asymptotics $-k^4/2 + O(1/n^2)$ and thus needs to be regularized. With zeta-function regularization $\zeta(0) = -1/2$ the energy shift is

$$\delta E_2^{\mathrm{reg}} = -\frac{3k^4}{8} + \frac{1}{8}\sum_n \left(-6k^4 + n^4 - (n^2+2k^2)n\sqrt{n^2-4k^2}\right). \qquad \text{(A.3)}$$

• **3-loop:** the naively expanded sum diverges as $\frac{9k^4}{8}n^2 + \frac{k^6}{4} + O(1/n^2)$, and needs to be regularized to give

$$\delta E_3^{\mathrm{reg}} = \frac{5k^6}{16} + \frac{1}{16}\sum_n \left(10k^6 - k^4n^2 + 2k^2n^4 - n^6 + (3k^4+n^4)n\sqrt{n^2-4k^2}\right). \qquad \text{(A.4)}$$

Given the relatively simple dependence on $\mathcal{J}$ of the string frequencies, one can compute the subtraction term, necessitated by zeta-function regularization in a closed form. Each of the frequencies is of the type $\sqrt{1+a/(\mathcal{J}^2+k^2)}$, which has an expansion around $\mathcal{J}=\infty$. Consider first the following term

$$\sqrt{1+\frac{a}{\mathcal{J}^2}} = \sum_{p=0}^{\infty}\binom{1/2}{p}\frac{a^p}{\mathcal{J}^{2p}}. \qquad \text{(A.5)}$$

Now, each a has an expansion in n, and we wish to determine the terms up to order $1/n^2$ for fixed value of p. Define

$$\begin{aligned} a_1 &= \frac{1}{4}\left(n+\sqrt{n^2-4k^2}\right)^2 + k^2 \\ a_2 &= n^2-k^2\,, \qquad a_3 = n^2+k^2\,, \qquad a_4 = n^2\,. \end{aligned} \qquad \text{(A.6)}$$

Then

$$\begin{aligned} &\delta E^{(n)}\sqrt{1+\frac{k^2}{\mathcal{J}^2}} = \qquad \text{(A.7)} \\ &= \sum_{p=0}^{\infty}\binom{1/2}{p}\frac{1}{\mathcal{J}^{2p}}\left\{2\left(k^2+\frac{(n+\sqrt{n^2-4k^2})^2}{4}\right)^p + 2(n^2-k^2)^p + 4(n^2+k^2)^p - 8n^{2p}\right\} \\ &= \sum_{p=0}^{\infty}\binom{1/2}{p}\frac{n^{2p}}{\mathcal{J}^{2p}}\left\{2\left(\frac{1+\sqrt{1-4k^2/n^2}}{2}\right)^p + 2(1-k^2/n^2)^p + 4(1+k^2/n^2)^p - 8\right\}. \end{aligned}$$

Then invoking

$$\left(1+\sqrt{1+x}\right)^p = 2^p + 2^p\sum_{q=1}^{\infty}\binom{p-q-1}{q-1}\frac{p}{q}\left(\frac{x}{4}\right)^q, \qquad \text{(A.8)}$$

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and the binomial theorem, we get

$$\delta E^{(n)}\sqrt{1+\frac{k^2}{\mathcal{J}^2}} = \tag{A.9}$$
$$= \sum_{p=0}^{\infty}\binom{1/2}{p}\frac{n^{2p}}{\mathcal{J}^{2p}}\left\{2\sum_{q=1}^{\infty}\binom{p-q-1}{q-1}\frac{p}{q}\left(-\frac{k^2}{n^2}\right)^q + 2\sum_{q=1}^{p}\binom{p}{q}(2+(-1)^q)\left(\frac{k^2}{n^2}\right)^q\right\}.$$

Further expanding $1/\sqrt{1+k^2/\mathcal{J}^2}$, the coefficient of the $1/\mathcal{J}^{2p}$ term is

$$(\delta E^{(n)})_p = 2\sum_{g=0}^{p}\binom{-1/2}{p-g}\binom{1/2}{g}k^{2(p-g)}n^{2g}\times \tag{A.10}$$
$$\times\left\{\sum_{q=1}^{\infty}\binom{g-q-1}{q-1}\frac{g}{q}\left(-\frac{k^2}{n^2}\right)^q + \sum_{q=1}^{g}\binom{g}{q}(2+(-1)^q)\left(\frac{k^2}{n^2}\right)^q\right\}.$$

Again this is an unpleasant-looking hypergeometric function. However, we only need to extract the coefficients up to the term $1/n$ of it, and in order to obtain the zero-point energy regularization, we only have to extract the coefficient of n^0. The subtraction term at order $1/\mathcal{J}^{2p}$ is

$$\mathfrak{S}_p = 2\sum_{g=0}^{p}\binom{-1/2}{p-g}\binom{1/2}{g}\sum_{q=1}^{g}k^{2(p+q-g)}n^{2(g-q)}(-1)^q\left\{\binom{g-q-1}{q-1}\frac{g}{q}+((-1)^q 2+1)\binom{g}{q}\right\}. \tag{A.11}$$

This agrees to three loops with the above explicitly obtained expressions. We can also determine the change to the zero-point energy, namely

$$(\delta E^{(0)})_p^{\mathrm{reg}} = \delta E_p^{(0)} - \frac{1}{2}\left(2k^{2p}\sum_{g=1}^{p}\binom{-1/2}{p-g}\binom{1/2}{g}(1+(-1)^g)\right). \tag{A.12}$$

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